

GEOMETRIC POINCARÉ LEMMA

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1. INTRODUCTION

The Poincaré Lemma for the de Rham complex $\Omega(U) = \oplus_{k=0}^n \Omega^k(U)$ of smooth differential forms defined on contractible open subsets U of \mathbb{R}^n is a foundational result with applications from topology to physics: *Every closed differential form $\omega \in \Omega_k(U)$ is exact.* That is, if $d\omega = 0$ where $\omega \in \Omega_k(U)$, then there exists $\eta \in \Omega_{k-1}(U)$ such that $\omega = d\eta$. Such forms η are found through a homotopy operator A acting on forms where $\eta = A\omega$. That such an operator A exists is a remarkable and powerful feature of the de Rham complex.

A homotopy operator K on the chain complex of polyhedral k -chains is easy to construct using the classical cone construction from topology, and works quite well for the category of polyhedral chains. One way to generalize Poincaré's Lemma is to introduce a topology on each vector space of polyhedral k -chains with the hope of extending K to a continuous homotopy operator on the chain complex of the completed spaces. For a coherent theory with broad application to domains of integration going far beyond polyhedral chains, we need the dual space to be an identifiable space of differential k -forms, and the dual operator $A\omega := \omega K$, which is necessarily continuous since K is continuous, to be a computable homotopy operator.

Whitney's Banach space of "sharp chains" [?] has no continuous boundary operator, and thus there is no meaningful cone operator in the sharp space. The cone operator K on polyhedral chains is not generally continuous in the Banach space of "flat chains", although it is continuous in the subspace of flat chains with finite mass, and this has been useful as a tool for solving a special case of Plateau's problem for integral currents ([?], see also [?]). Our efforts to solve the Plateau problem in full generality [?] led to the results in this paper.

Our main result is a geometric Poincaré Lemma for a certain differential chain complex (to be defined below) of topological vector spaces $\hat{\mathcal{B}}_k(U_1)$ of "differential k -chains" in an open set U_1 that

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is contractible in an open set U_2 with $U_1 \subset U_2 \subset \mathbb{R}^n$. That is, there exists $F : [0, 1] \times U_1 \rightarrow U_2$ with $F(0, p) = p$ and $F(1; p) = c$ for some constant $c \in U_2$.

Every cycle $J \in \hat{\mathcal{B}}_k(U_1)$ is a boundary in $\hat{\mathcal{B}}_k(U_2)$ for all $1 \leq k \leq n - 1$.

That is, if $J \in \hat{\mathcal{B}}_k(U_1)$ satisfies $\partial J = 0$, there exists $K \in \hat{\mathcal{B}}_{k+1}(U_2)$ with $\partial K = J$. (See Theorem 3.1.2 in §3.)

As a corollary, we obtain the following dual result. Let $\mathcal{B}_k(U)$ be the Frechét space of differential forms defined on U , each with uniform bounds on each of its directional derivatives. Then $\mathcal{B}_k(U)$ is the topological dual of $\hat{\mathcal{B}}_k(U)$ (see [?] Theorem 2.12.8).

Every closed form $\omega \in \mathcal{B}_k(U_2)$ is exact in $\mathcal{B}_{k-1}(U_1)$, for all $1 \leq k \leq n - 1$.

That is, if $\omega \in \mathcal{B}_k(U_2)$ satisfies $d\omega = 0$, there exists $\eta \in \mathcal{B}_{k-1}(U_1)$ with $\omega|_{U_1} = d\eta$.

There are three primary reasons for calling our result a “geometric Poincaré Lemma”. First of all, the topology of $\hat{\mathcal{B}}_k(U)$ is defined constructively, and is not simply the abstract dual of the space of currents $\mathcal{B}_k(U)'$. Similarly, the cone operator K is defined geometrically. Finally, Dirac chains are dense in $\hat{\mathcal{B}}_k(U)$, yielding a discrete and computable version of the geometric Poincaré’s lemma and its cone operator K . The operator K restricts to the classical cone operator on polyhedral chains, and thus our theorem is a generalization of the classical version of Poincaré’s Lemma for chains.

Applications presented in this paper include a broad generalization of the Intermediate Value Theorem to arbitrary dimension and codimension (see Theorem 4.1.1 and Figure 1), a new approach to homology theory with potential extensions to non-manifolds [?]. H. Pugh uses results of this paper to find generalizations of the Cauchy integral theorems [?], and, in a sequel [?], the author uses the geometric Poincaré Lemma to provide a general solution to Plateau’s problem.

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2. PRELIMINARIES

This work relies on new methods of calculus presented in [?]. In this preliminary section we recount the definition of the topological vector space $\hat{\mathcal{B}}(U)$ and its operators from [?] which we use below.

2.1. The Mackey topology on Dirac chains. Let $\mathcal{A}_k(U)$ be the vector space of *Dirac k -chains*, i.e., formal sums $\sum(p_i; \alpha_i)$ where $p_i \in U$ and $\alpha_i \in \Lambda_k(\mathbb{R}^n)$. We call $(p; \alpha)$ a *simple k -element* if α is a simple k -vector. Otherwise, $(p; \alpha)$ is called a *k -element*. Since $(\mathcal{A}_k(U), \mathcal{B}_k(U))$ is a dual pair, the Mackey topology τ_k is uniquely determined on $\mathcal{A}_k(U)$. (The Mackey topology τ_k is the finest topology μ_k on $\mathcal{A}_k(U)$ such that $(\mathcal{A}_k(U), \mu_k)' = \mathcal{B}_k(U)$.) This process may be mimicked for

any subspace of forms, e.g., $\mathcal{D}_k(U)$, the space of smooth functions with compact support in U , but $\mathcal{B}_k(U)$ is especially nice to work with because of its useful algebra of continuous operators, and algebraic features of its topology. The complex $\hat{\mathcal{B}}_k(U)$, which can also be defined for open subsets of Riemannian manifolds, is intrinsic in the following sense recently announced in [?]:

Theorem. The topology $\tau_k(U)$ is the finest topology in the collection $\{\mu\}$ of locally convex Hausdorff topologies on Dirac chains $\mathcal{A}_k(U)$ satisfying three axioms:

- (a) The topological vector space $(\mathcal{A}_k(U), \mu)$ is bornological;
- (b) $K^0 = \{(p; \alpha) \in \mathcal{A}_k(U) : \|\alpha\| = 1\}$ is bounded where $\|\alpha\|$ is the mass norm of α ;
- (c) The linear map $P_v : (\mathcal{A}_k(U), \mu) \rightarrow \overline{(\mathcal{A}_k(U), \mu)}$ determined by $P_v(p; \alpha) := \lim_{t \rightarrow 0} (p + v; \alpha/t) - (p; \alpha/t)$ is well-defined and uniformly bounded for all unit vectors $v \in \mathbb{R}^n$.

The first two properties are essential to analysis in this space, the third suffices for the Lie derivative of differential forms to be a bounded operator. It is shown in [?] that the Mackey topology τ_k coincides with the constructive definition of τ_k of [?], which we next recall.

2.2. Constructive definition of the B^r norms.

2.2.1. Mass norm. An inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n determines the mass norm on $\Lambda_k(U)$ as follows: Let $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)$. The *mass* of a simple k -vector $\alpha = v_1 \wedge \cdots \wedge v_k$ is defined by $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. The mass of a k -vector α is $\|\alpha\| := \inf \left\{ \sum_{j=1}^N \|\alpha_j\| : \alpha_j \text{ are simple, } \alpha = \sum_{j=1}^N \alpha_j \right\}$. Define the *mass* of a k -element $(p; \alpha)$ by $\|(p; \alpha)\|_{B^0} := \|\alpha\|$. Mass is a norm on the subspace of Dirac k -chains supported in p , since that subspace is isomorphic to the exterior algebra $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda_k(\mathbb{R}^n)$ for which mass is a norm (see [?], p 38-39). The *mass* of a Dirac k -chain $A = \sum_{i=1}^m (p_i; \alpha_i) \in \mathcal{A}_k(U)$ is given by

$$\|A\|_{B^0} := \sum_{i=1}^m \|(p_i; \alpha_i)\|_{B^0}.$$

If a different inner product is chosen, the resulting masses of Dirac chains are topologically equivalent. It is straightforward to show that $\|\cdot\|_{B^0}$ is a norm on $\mathcal{A}_k(U)$.

2.2.2. Difference chains and the B^r norm on chains. Given a k -element $(p; \alpha)$ with $p \in U$ and $u \in \mathbb{R}^n$, let $T_u(p; \alpha) := (p + u; \alpha)$ be translation through u , and $\Delta_u(p; \alpha) := (T_u - I)(p; \alpha)$. Let $S^j = S^j(\mathbb{R}^n)$ be the j -th *symmetric power* of the symmetric algebra $S(\mathbb{R}^n)$. Denote the symmetric product in the symmetric algebra $S(\mathbb{R}^n)$ by \circ . Let $\sigma = \sigma^j = u_j \circ \cdots \circ u_1 \in S^j$ with $u_i \in \mathbb{R}^n, i = 1, \dots, j$. Recursively define $\Delta_{u \circ \sigma^j}(p; \alpha) := (T_u - I)(\Delta_{\sigma^j}(p; \alpha))$. Let $\|\sigma\| := \|u_1\| \cdots \|u_j\|$ and

$|\Delta_{\sigma^j}(p; \alpha)|_{B^j} := \|\sigma\| \|\alpha\|$. Let $\Delta_{\sigma^0}(p; \alpha) := (p; \alpha)$, to keep the notation consistent. We say $\Delta_{\sigma^j}(p; \alpha)$ is *inside* U if the convex hull of $\text{supp}(\Delta_{\sigma^j}(p; \alpha))$ is a subset of U .

Definition 2.2.1. For $A \in \mathcal{A}_k(U)$ and $r \geq 0$, define the seminorm

$$\|A\|_{B^r, U} := \inf \left\{ \sum_{j=0}^r \sum_{i=1}^{m_j} \|\sigma_{j_i}\| \|\alpha_{j_i}\| : A = \sum_{j=0}^r \sum_{i=1}^{m_j} \Delta_{\sigma_{j_i}^j}(p_{j_i}; \alpha_{j_i}) \text{ where } \Delta_{\sigma_{j_i}^j}(p_{j_i}; \alpha_{j_i}) \text{ is inside } U \right\}.$$

For simplicity, we often write $\|A\|_{B^r} = \|A\|_{B^r, U}$ if U is understood. It is easy to see that the B^r norms on Dirac chains are decreasing as r increases.

It is shown in [?] (Theorem 2.6.1) that $\|\cdot\|_{B^r}$ is a norm on the free space of Dirac k -chains $\mathcal{A}_k(U)$ called the B^r norm. Let $\hat{\mathcal{B}}_k^r(U)$ be the Banach space obtained upon completion of $\mathcal{A}_k(U)$ with the B^r norm. Elements of $\hat{\mathcal{B}}_k^r(U)$, $0 \leq r \leq \infty$, are called *differential¹ k -chains of class B^r in U* . Let $\hat{\mathcal{B}}_k(U) = \hat{\mathcal{B}}_k^\infty(U) := \varinjlim \hat{\mathcal{B}}_k^r(U)$ be the inductive limit as $r \rightarrow \infty$, endowed with the inductive limit topology. It is shown in [?] that $\overline{\hat{\mathcal{B}}_k(U)} \cong \overline{(\mathcal{A}_k(U), \tau_k)}$. Therefore, the inductive limit topology coincides with the Mackey topology τ_k .

Let $\mathcal{B}_k^0(U)$ be the Banach space of bounded measurable k -forms, $\mathcal{B}_k^1(U)$ the Banach space of Lipschitz k -forms, and for each $r > 1$, $\mathcal{B}_k^r(U)$ be the Banach space of differential k -forms, each with bounds on each of the s -th order directional derivatives for $0 \leq s \leq r-1$ and the r -th derivatives satisfy a Lipschitz condition. Denote the resulting norm by $\|\cdot\|_{B^r}$. Elements of $\mathcal{B}_k^r(U)$ are called *differential k -forms of class B^r in U* . The natural inclusions $\hat{\mathcal{B}}_k^r(U_1) \hookrightarrow \hat{\mathcal{B}}_k^r(U_2)$ are continuous for all open $U_1 \subset U_2 \subset \mathbb{R}^n$. Let $\mathcal{B}_k(U) = \mathcal{B}_k^\infty(U) = \varprojlim \mathcal{B}_k^r(U)$ be the projective limit as $r \rightarrow \infty$, endowed with the Frechét topology. If $U_1 \subset U_2$, the natural restrictions $\mathcal{B}_k^r(U_2) \rightarrow \mathcal{B}_k^r(U_1)$ are continuous, but they are not surjective unless U_1 is a regular open subset, that is, U_1 equals the interior of its closure. It is shown in [?] Theorem 2.8.2 that $(\hat{\mathcal{B}}_k^r(U))' \cong \mathcal{B}_k^r(U)$, and the integral pairing $f : \hat{\mathcal{B}}_k^r(U) \otimes \mathcal{B}_k^r(U) \rightarrow \mathbb{R}$ where $J \otimes \omega \mapsto \omega(J)$ is nondegenerate. Let $f_J \omega := \omega(J)$ for all $J \in \hat{\mathcal{B}}_k^r(U)$ and $\omega \in \mathcal{B}_k^r(U)$.

If J is nonzero then its support $\text{supp}(J)$ is a uniquely determined nonempty set (see [?] Theorem 6.1.3). The *support* of a nonzero k -chain $J \in \hat{\mathcal{B}}_k(U)$ is the smallest closed subset $E \subset U$ such that $f_J \omega = 0$ for all smooth ω with compact support disjoint from E (see [?] Theorem 6.2.2).

2.3. Operators. Let $\mathcal{B}(U) := \oplus_{k=0}^n \mathcal{B}_k(U)$. Since $\hat{\mathcal{B}}_k(U)$ is a subspace of currents $\mathcal{B}_k(U)'$ which is proper if $U = \mathbb{R}^n$ (see [?]), a natural question arises: If $T : \mathcal{B}(U_1) \rightarrow \mathcal{B}(U_2)$ is a continuous operator on forms, we know its dual $S(\omega) := \omega(T)$ is continuous on currents $S : \mathcal{B}(U_2)' \rightarrow \mathcal{B}(U_1)'$.

¹The adjective “differential” here does not indicate that the chains are in any way smooth, only that they are closed under the topological dual to the Lie derivative operator of differential forms.

Is $S : \hat{\mathcal{B}}(U_2) \rightarrow \hat{\mathcal{B}}(U_1)$ closed? The constructive definition of the Mackey topology τ_k is often useful for answering such questions as seen in the following Lemma.

Lemma 2.3.1. *Fix $0 \leq s \leq r$ and $0 \leq k \leq n$. If $T : \mathcal{A}(U_1) \rightarrow \mathcal{A}(U_2)$ is a graded operator satisfying*

$$\|T(\Delta_{\sigma^j}(p; \alpha))\|_{B^r} \leq C \|\sigma\| \alpha\|$$

for some constant $C > 0$ and all j -difference k -chains $\Delta_{\sigma^j}(p; \alpha)$ inside U_1 with $\sigma^j \in S^j(\mathbb{R}^n)$, $0 \leq j \leq s$, $\alpha \in \Lambda_k(\mathbb{R}^n)$, then $\|T(A)\|_{B^r} \leq C \|A\|_{B^s}$ for all $A \in \mathcal{A}_k(U_1)$.

See [?], Theorem 2.7.3.

Define $E_v(p; \alpha) = (p; v \wedge \alpha)$. Suppose $\beta = v_1 \wedge \cdots \wedge v_s$ is simple. Define $E_\beta = E_{v_s} \circ \cdots \circ E_{v_1}$ and $i_\beta = i_{v_1} \circ \cdots \circ i_{v_s}$.

Theorem 2.3.2. *Fix $\beta \in \Lambda_s(\mathbb{R}^n)$. Then $E_\beta : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+s}^r(U)$ and $i_\beta : \mathcal{B}_k^{r+s}(U) \rightarrow \mathcal{B}_k^r(U)$ are continuous graded operators with $\int_{E_\beta J} \omega = \int_J i_\beta \omega$. Furthermore, $\|E_\beta(J)\|_{B^r} \leq \|\beta\| \|J\|_{B^r}$.*

See [?] Theorem 3.1.3 for a proof.

Boundary $\partial = \partial_k$ coincides with the classical boundary operator ∂_k on polyhedral k -chains, which are dense in $\hat{\mathcal{B}}_k(U)$, although a direct definition using “higher order Dirac chains” is provided in [?] §4.

Theorem 2.3.3. *[General Stokes’ Theorem] The boundary operator on differential chains $\partial : \hat{\mathcal{B}}_k^r \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}$ is continuous, $\partial \circ \partial = 0$, and $\|\partial J\|_{B^{r+1}} \leq kn \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r$. Furthermore, exterior derivative $d : \mathcal{B}_{k-1}^{r+1}(U) \rightarrow \mathcal{B}_k^r(U)$ is continuous. If $\omega \in \mathcal{B}_{k-1}^r$ is a differential form and $J \in \hat{\mathcal{B}}_k^{r-1}$ is a differential chain, then*

$$\int_{\partial J} \omega = \int_J d\omega.$$

See [?] (Theorems 4.1.3 and 4.1.6).

We say a differential k -chain $\gamma \in \hat{\mathcal{B}}_k(U)$ is a *differential k -cycle* if $\partial\gamma = 0$.

Differential 0-forms f determine functions $f : U \rightarrow \mathbb{R}$ by $f(p) := f(p; 1)$. We say that the function f is of class B^r if the 0-form $f \in \mathcal{B}_0^r$. In (see [?] §2.9) we prove that f is of class B^r if and only if f is of class $C^{r-1+Lip}$, i.e., f is $(r-1)$ -times continuously differentiable, and its $(r-1)$ -st directional derivatives are Lipschitz. For each $f : U \rightarrow \mathbb{R}$ of class B^r define a map $m_f : \mathcal{A}_k(U) \rightarrow \mathcal{A}_k(U)$ by $m_f(p; \alpha) := (p; f(p)\alpha)$ for simple k -elements $(p; \alpha)$, $p \in U$ and linearly extend.

Theorem 2.3.4. *If f is of class B^r , then the linear map $m_f : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^r(U)$ is well-defined and continuous. The topological vector space of differential chains $\hat{\mathcal{B}}^r(U)$ is therefore a bigraded module over the ring of operators $\mathcal{B}_0^r(U)$. Furthermore, $\|m_f A\|_{B^r} \leq nr\|f\|_{B^r}\|A\|_{B^r}$ and $\partial m_f = m_f \partial + m_{df}$.*

For a proof see [?] Theorems 5.1.2 and 5.1.3.

2.4. Mappings and the pushforward operator. Suppose $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ are open and $F : U_1 \rightarrow U_2$ is a differentiable map. For $p \in U_1$, define *linear pushforward* $F_{p*}(v_1 \wedge \cdots \wedge v_k) := DF_p(v_1) \wedge \cdots \wedge DF_p(v_k)$ where DF_p is the total derivative of F at p .

Define $F_*(p; \alpha) := (F(p), F_{p*}\alpha)$ for all simple k -elements $(p; \alpha)$ and extend to a linear map $F_* : \mathcal{A}_k(U_1) \rightarrow \mathcal{A}_k(U_2)$ called *pushforward*.

Definition 2.4.1.

Let $\mathcal{M}^r(U, \mathbb{R}^m)$ be the vector space of differentiable maps $F : U \rightarrow \mathbb{R}^m$ so that the directional derivatives $L_{e_j} F_i$ of its coordinate functions F_i are of class B^{r-1} , for $r \geq 1$. Define the seminorm $|F|_{D^r, U} := \max_{i,j} \{\|L_{e_j} F_i\|_{B^{r-1}, U}\}$.

We write $|F|_{D^r} = |F|_{D^r, U}$ when U is understood. It is not hard to show that $|\cdot|_{D^r}$ is a seminorm, but not a norm, on the vector space $\mathcal{M}^r(U, \mathbb{R}^m)$. Let $\mathcal{M}^\infty(U, \mathbb{R}^m)$ be the projective limit of $\mathcal{M}^r(U, \mathbb{R}^m)$. Let $\mathcal{M}^r(U_1, U_2)$ denote the subset $\{F \in \mathcal{M}^r(U_1, \mathbb{R}^m) : F(U_1) \subset U_2 \subset \mathbb{R}^m\}$ where U_2 is open for all $0 \leq r \leq \infty$. If $U \subset \mathbb{R}^n$ is a regular open set, we can similarly define $\mathcal{M}^r(\bar{U}, \mathbb{R}^m)$ as the space of maps $F : \bar{U} \rightarrow \mathbb{R}^m$ which extend to maps defined in a neighborhood of \bar{U} .

Theorem 2.4.2. *If $F \in \mathcal{M}^r(U_1, U_2)$, then*

$$\|F_*(A)\|_{B^r, U_2} \leq mn \max\{1, r|F|_{D^r, U_1}\} \|A\|_{B^r, U_1}$$

for all $A \in \mathcal{A}_0(U_1)$ and all $r \geq 0$. and thus determines a continuous linear map $F_ : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ with $\int_{F_* J} \omega = \int_J F_* \omega$ for all $J \in \hat{\mathcal{B}}_k^r(U_1)$ and $\omega \in \mathcal{B}_k^r(U_2)$. Furthermore, $\partial \circ F_* = F_* \circ \partial$.*

For a proof see [?] Theorems 7.5.4 and 7.5.5.

Theorem 2.4.3. *Any affine k -cell τ in U is represented by a unique differential k -chain $\tilde{\tau} \in \hat{\mathcal{B}}_k(U)$ such that $\int_{\tilde{\tau}} \omega = \int_\tau \omega$ for all $\omega \in \mathcal{B}_k^1(U)$.*

For a proof see [?], Theorem 2.11.2.

2.5. Cartesian wedge product. Suppose $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ are open. Let $\iota_1 : U_1 \rightarrow U_1 \times U_2$ and $\iota_2 : U_2 \rightarrow U_1 \times U_2$ be the inclusions $\iota_1(p) = (p, 0)$ and $\iota_2(q) = (0, q)$. Let $\pi_1 : U_1 \otimes U_2 \rightarrow U_1$ and $\pi_2 : U_1 \otimes U_2 \rightarrow U_2$ be the projections $\pi_i(p_1, p_2) = p_i$, $i = 1, 2$. Let $(p; \alpha) \in \mathcal{A}_k(U_1)$ and $(q; \beta) \in \mathcal{A}_\ell(U_2)$. Define $\times : \mathcal{A}_k(U_1) \times \mathcal{A}_\ell(U_2) \rightarrow \mathcal{A}_{k+\ell}(U_1 \times U_2)$ by $\times((p; \alpha), (q; \beta)) := ((p, q); \iota_{1*}\alpha \wedge \iota_{2*}\beta)$ where $(p; \alpha)$ and $(q; \beta)$ are k - and ℓ -elements, respectively, and extend bilinearly. We call $P \times Q = \times(P, Q)$ the *Cartesian wedge product*² of P and Q . Cartesian wedge product of Dirac chains is associative since wedge product is associative, but it is not graded commutative since Cartesian product is not graded commutative. The next result shows that Cartesian wedge product is continuous.

Theorem 2.5.1. *Cartesian wedge product $\times : \hat{\mathcal{B}}_k^r(U_1) \times \hat{\mathcal{B}}_\ell^s(U_2) \rightarrow \hat{\mathcal{B}}_{k+\ell}^{r+s}(U_1 \times U_2)$ is associative, bilinear and continuous for all open sets $U_1 \subset \mathbb{R}^n, U_2 \subset \mathbb{R}^m$ and satisfies*

- (a) $\|J \times K\|_{B^{r+s}, U_1 \times U_2} \leq \|J\|_{B^r, U_1} \|K\|_{B^s, U_2}$
- (b) $\|\widetilde{(a, b)} \times J\|_{B^r, \mathbb{R} \times U_1} \leq |b - a| \|J\|_{B^r, U_1}$ where $\widetilde{(a, b)}$ is the 1-chain representing the interval (a, b) .
- (c) $\partial(J \times K) = (\partial J) \times K + (-1)^k J \times (\partial K)$.

See [?] Theorem 10.1.3.

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3.1. Chain homotopy. Since $\partial_{k-1} \circ \partial_k = 0$,

$$(1) \quad \hat{\mathcal{B}}_n^s(U) \xrightarrow{\partial_n} \hat{\mathcal{B}}_{n-1}^{s+1}(U) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} \hat{\mathcal{B}}_0^{s+n}(U) \xrightarrow{\partial} \{0\}.$$

is a bigraded chain complex. Letting $s \rightarrow \infty$, the inductive limits

$$(2) \quad \hat{\mathcal{B}}_n(U) \xrightarrow{\partial_n} \hat{\mathcal{B}}_{n-1}(U) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} \hat{\mathcal{B}}_0(U) \xrightarrow{\partial} \{0\}.$$

form a chain complex since ∂_k is well-defined and continuous on the inductive limits.

A collection of maps $S_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ is a *graded map of chain complexes* if $S_{k-1} \circ \partial_k = \partial_k \circ S_k$.

Definition 3.1.1. For $U \subset \mathbb{R}^n$ open, the k -th “differential homology” group of the chain complex $(\hat{\mathcal{B}}_k(U), \partial_k)$ is

$$\hat{H}_k(U) := \frac{\ker \partial_k}{\text{im } \partial_{k+1}}.$$

²By the universal property of tensor product, \times factors through a continuous linear map *cross product* $\tilde{\times} : \hat{\mathcal{B}}_j(U_1) \otimes \hat{\mathcal{B}}_k(U_2) \rightarrow \hat{\mathcal{B}}_{j+k}(U_1 \times U_2)$. This is closely related to the classical definition of *cross product* on simplicial chains ([?], p. 278)

Let U_ϵ be the ϵ -neighborhood of a smoothly embedded m -manifold M in \mathbb{R}^n . Define³

$$\hat{H}_k(M) := \varinjlim \hat{H}_k(U_\epsilon)$$

where the inductive limit is taken as $\epsilon \rightarrow 0$.

If $F_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ is a graded map of chain complexes, then F_k induces a well-defined map $\hat{H}_k(F_k) : \hat{H}_k(U_1) \rightarrow \hat{H}_k(U_2)$ with $\hat{H}_k(F_k)[J] := \hat{H}_k[F_k J]$ since F_k commutes with ∂ .

Two graded maps of complexes $F_k, G_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ are *chain homotopic* through a family of maps $K_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_{k+1}^r$ if $G_k - F_k = \partial K_k + K_{k-1} \partial$. It is a standard result that if two maps of complexes F_k, G_k are chain homotopic, then $\hat{H}_k(F_k) = \hat{H}_k(G_k)$.

Two maps $F_0, F_1 : U_1 \subset \mathbb{R}^n \rightarrow U_2 \subset \mathbb{R}^m$ are *B^r homotopic* if there exists $F : [0, 1] \times U_1 \rightarrow U_2$ with $F(0, p) = F_0(p)$ and $F(1, p) = F_1(p)$ and $F \in \mathcal{M}^r([0, 1] \times U_1 \rightarrow U_2)$. We say U_1 is *B^r contractible* in U_2 if $U_1 \subset U_2 \subset \mathbb{R}^n$ are open and the inclusion $U_1 \hookrightarrow U_2$ with $p \mapsto p \in U_1$ is B^r homotopic to a constant map $U_1 \rightarrow U_2$ with $p \mapsto c \in U_2$.

We may now state our main result.

Theorem 3.1.2. [*Poincaré Lemma for Differential Chains*] Let $U_1 \subset U_2 \subset \mathbb{R}^n$ be open subsets where U_1 is B^r contractible in U_2 . Then for $J \in \hat{\mathcal{B}}_k^r(U_1)$ with $\partial J = 0$, there exists $C \in \hat{\mathcal{B}}_{k+1}^r(U_2)$ with $\partial C = J$ for all $1 \leq k \leq n$ and $1 \leq r \leq \infty$.

In order to prove this we first establish Corollary 3.1.6 which says that pushforwards through homotopic maps are homotopic as maps of complexes.

Let \tilde{I} denote the 1-chain representing the interval $[0, 1] \subset \mathbb{R}$ (see Theorem 2.4.3), and $L = L_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r((0, 1) \times U_1)$ be Cartesian wedge product $L_k(J) := \tilde{I} \times J$ (see Theorem 2.5.1). Recall that $(t; 1)$ is the simple unit 0-element supported in $t \in \mathbb{R}^1$.

Lemma 3.1.3. $\|LJ\|_{B^r} \leq \|J\|_{B^r}$ and $(\partial L + L\partial)(J) = (1; 1) \times J - (0; 1) \times J$.

Proof. According to the Test Lemma, the inequality reduces to showing $\|\tilde{I} \times \Delta_{\sigma^j}(p; \alpha)\|_{B^r} \leq \|\sigma\|\alpha\|$ for all $0 \leq j \leq r$. Using Theorem 2.5.1 we know $\|\tilde{I} \times \Delta_{\sigma^j}(p; \alpha)\|_{B^r} \leq \|\Delta_{\sigma^j}(p; \alpha)\|_{B^r} \leq \|\sigma\|\|\alpha\|$. \square

Let $F : [0, 1] \times U_1 \rightarrow U_2$ be an element of $\mathcal{M}^r([0, 1] \times U_1 \rightarrow U_2)$ with $F(0, p) = f_0(p)$ and $F(1, p) = f_1(p)$. Then $F_* : \hat{\mathcal{B}}_k^r((0, 1) \times U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ is a continuous linear map. We remark that $\widetilde{[0, 1]} \times (p; 1) \in$

³In a sequel in preparation, M.W. Hirsch and the author show that $\hat{H}_k(M)$ coincides with singular homology. They are developing a homology theory $\hat{H}_k(J)$ for arbitrary differential ℓ -chains $J \in \hat{\mathcal{B}}_\ell(U)$ $0 \leq \ell \leq n$, but it is not yet clear which axioms of homology such a theory would satisfy.

$\hat{\mathcal{B}}_k^r((0, 1) \times U_1)$ for all $p \in U_1$, even though $[0, 1]$ is closed (see [?] for a full discussion about chains in open sets.). Let $K = K_k := F_* L_k : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_{k+1}^{r-1}(U_2)$ for $r \geq 0$.

Theorem 3.1.4. *If $F \in \mathcal{M}^r([0, 1] \times U_1 \rightarrow U_2)$, then $K = F_* L$ extends to a continuous linear map $K : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U_2)$ satisfying $\|KJ\|_{B^r} \leq mn \max\{1, r|F|_{D^r}\} \|J\|_{B^r}$ for all differential chains $J \in \hat{\mathcal{B}}_k^r(U_1)$ for all $0 \leq k \leq n-1$.*

Proof. $\|KJ\|_{B^r} = \|F_* L J\|_{B^r} \leq mn \max\{1, r|F|_{D^r}\} \|LJ\|_{B^r} \leq mn \max\{1, r|F|_{D^r}\} \|J\|_{B^r}$. \square

Theorem 3.1.5. *If $f_0, f_1 : U_1 \subset \mathbb{R}^n \rightarrow U_2 \subset \mathbb{R}^m$ are B^r homotopic, the maps of chain complexes $f_{1*}, f_{0*} : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ are chain homotopic through the family of maps $\{K_k\}$. That is,*

$$\partial_{k+1} K_k + K_{k+1} \partial_k = f_{1*} - f_{0*}$$

for all $0 \leq k \leq n-1$.

Proof. By Theorems 2.4.2 and 2.5.1

$$\begin{aligned} (\partial K + K \partial)(J) &= (\partial F_* L + F_* L \partial)(J) = F_*(\partial L + L \partial)(J) \\ &= F_*(\partial(\tilde{I} \times J) + \tilde{I} \times \partial J) \\ &= F_*((\partial \tilde{I}) \times J - \tilde{I} \times \partial J + \tilde{I} \times \partial J) \\ &= F_*((\partial \tilde{I}) \times J) \\ &= F_*((1; 1) \times J - (0; 1) \times J) \\ &= (f_{1*} - f_{0*})(J). \end{aligned}$$

\square

If $k = 0$, $f_0 = c$ and $f_1 = I$, then $f_{1*} - f_{0*} = I_*$, but $\partial K + K \partial = \partial K \neq I$, so the theorem fails for $k = 0$.

Corollary 3.1.6. *Pushforwards through homotopic maps act identically on the homologies. That is, $H_k(f_{0*}) = H_k(f_{1*})$. In particular, on a contractible domain, $H_k(I_*) = H_k(c_*)$, where c is the constant map.*

Proof of the Geometric Poincaré Lemma: $\hat{H}_k(c_*) = 0$ and thus $\hat{H}_k(I_*) = 0$. But this implies $\hat{H}_k(U_1) = 0$. That is, $\text{im } \partial = \ker \partial$ which is what we wanted to prove.

The cone over a differential cycle is unique up to addition of a differential chain boundary since $J = \partial(KJ + \partial C)$. The chain ∂C is a *gauge choice*. In codimension one, $L = 0$.

3.2. Poincaré Lemma for forms. Suppose $F \in \mathcal{M}^r([0, 1] \times U_1 \rightarrow U_2)$. Let $F_t : U_1 \rightarrow U_2$ be given by $F_t(p) := F(t, p)$ and $K_k = F_* L_k$. Define $A_k \omega := \omega K_k$.

Theorem 3.2.1. $A_k : \mathcal{B}_{k+1}(U_2) \rightarrow \mathcal{B}_k(U_1)$ is a continuous linear map for all $1 \leq k \leq n-1$ and satisfies

- (a) $dA_k + A_{k+1}d = F_1^* - F_0^*$;
- (b) $\|A\omega\|_{B^r} \leq mn \max\{1, r|F|_{D^r}\} \|\omega\|_{B^r}$ for all $\omega \in \hat{\mathcal{B}}_k^r(U_2)$;
- (c) $(A\omega)(p; \alpha) = \int_0^1 i_{\partial/\partial t} \omega(F_t(p); \alpha) dt$

Proof. (a) and (b) follow immediately from Lemma 3.1.3 and Theorem 2.4.2.

(c): Recall from Theorem 2.4.3 that if $\tilde{\gamma}$ represents $[0, 1]$ in $\hat{\mathcal{B}}_1^r(\mathbb{R}^1)$, and $f dt \in \mathcal{B}_1^r(\mathbb{R}^1)$ is a 1-form, then $f_{\tilde{\gamma}} f dt = \int_0^1 f(t) dt$. Let $(p; \alpha)$ be a simple k -element with $p \in U_1$. Let $\tilde{\gamma}_p$ be the k -chain representing the interval $[0, 1] \times \{p\} \subset [0, 1] \times U_1$. If $\omega \in \mathcal{B}_{k+1}^r(U_2)$, then

$$\begin{aligned}
 A\omega(p; \alpha) &= \int_{K(p; \alpha)} \omega = \int_{F_* L(p; \alpha)} \omega = \int_{\tilde{I} \times (p; \alpha)} F^* \omega && \text{by Theorem 2.4.2} \\
 &= (-1)^k \int_{(p; \alpha) \times \tilde{I}} F^* \omega && \text{by antisymmetry of } \wedge \\
 &= (-1)^k \int_{E_\alpha \tilde{\gamma}_p} F^* \omega && \text{by the definition of } \times \\
 &= (-1)^k \int_{\tilde{\gamma}_p} i_\alpha F^* \omega && \text{by Theorem 2.3.2} \\
 &= \int_{\tilde{\gamma}_p} i_{\partial/\partial t} i_\alpha F^* \omega \wedge dt && \text{by the definition of interior product} \\
 &= \int_0^1 i_{\partial/\partial t} i_\alpha F^* \omega((t, p); 1) dt \\
 &= \int_0^1 i_{\partial/\partial t} \omega(F_t(p); \alpha) dt
 \end{aligned}$$

□

Remarks 3.2.2.

- If $k = 0$ and $\omega \in \mathcal{B}_1^r(U_2)$, then $A\omega$ is a 0-form, i.e., a function, and $A\omega(p) = A\omega(p; 1) = \int_0^1 i_{\partial/\partial t} \omega \circ F_t(p) dt$, the classical formula for the homotopy operator of functions.
- If $k = n-1$ then $\omega \in \mathcal{B}_n^r(U_2)$, and thus $\omega = g dV$ where $g \in \mathcal{B}_0^r(U_2)$. It follows that $A\omega = A g dV$ is an $(n-1)$ -form and $A(g dV)(p; \alpha) = \int_0^1 i_{\partial/\partial t} g dV(F_t(p); \alpha) dt = \int_0^1 g(F_t(p)) dt$.

We immediately deduce:

Corollary 3.2.3 (Poincaré Lemma for forms). *Let $U_1 \subset U_2 \subset \mathbb{R}^n$ be open and U_1 be B^r contractible in U_2 , and $2 \leq k \leq n-1$. If $\omega \in \mathcal{B}_k^r(U_2)$ satisfies $d\omega = 0$, there exists $\eta \in \mathcal{B}_{k-1}^r(U_1)$ such that $\omega|_{U_1} = d\eta$.*

Proof. Let $F : I \times U_1 \rightarrow U_2$ be the contraction. Then $F_{0*} = I_*$ and $F_{1*} = 0$. Set $\eta = A\omega$ and let $p \in U_1$. Applying Stokes' Theorem 2.3.3 twice, and Theorem 3.1.5 we have

$$d\eta(p; \alpha) = d(\omega K)(p; \alpha) = \int_{K\partial(p; \alpha)} \omega = \int_{(p; \alpha)} \omega - \int_{\partial K(p; \alpha)} \omega = \omega(p; \alpha).$$

□

4. APPLICATIONS

Lemma 4.0.4. *Let $n > 0$. There does not exist a nonzero differential n -cycle in a contractible open set of a smooth n -manifold M .*

Proof. If $\partial J = 0$, then $K\partial J = 0$. However, $KJ = 0$ since every $(n+1)$ -chain in \mathbb{R}^n is degenerate. Thus

$$J = \partial KJ + K\partial J = 0.$$

□

4.1. Generalization of the Intermediate Value Theorem.

Theorem 4.1.1. (General Intermediate Value Theorem) *Suppose $G : U_1 \subset \mathbb{R}^n \rightarrow U_2 \subset \mathbb{R}^m$ is an element of $\mathcal{M}^r(U_1, U_2)$ where $1 \leq n \leq m$ and $1 \leq r \leq \infty$. Let $J \in \hat{\mathcal{B}}_n^r(U_1)$ and $K \in \hat{\mathcal{B}}_n^r(U_2)$. Then*

$$G_*(\partial J) = \partial K \iff G_*J = K.$$

Proof. Suppose $G_*(\partial J) = \partial K$. Since pushforward commutes with boundary, $\partial(G_*J - K) = 0$. By Lemma 4.0.4 it follows that $G_*J = K$. The converse is immediate since the boundary operator is continuous and commutes with pushforward⁴. □

This significantly strengthens the conclusion of the classical result in topology: If $G : B^n \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous map whose restriction to the boundary of the ball B^n is the identity map, then the

⁴M.W. Hirsch helped the author clarify and simplify the original proof of the general intermediate value theorem announced as Corollary 22.8 in [?].

image of g contains all of B^n . Our result shows that if G is Lipschitz, then $G_*\widetilde{B}^n = \widetilde{B}^n$. For $n = m = 1$, this generalizes the intermediate value theorem (see Figure 1).

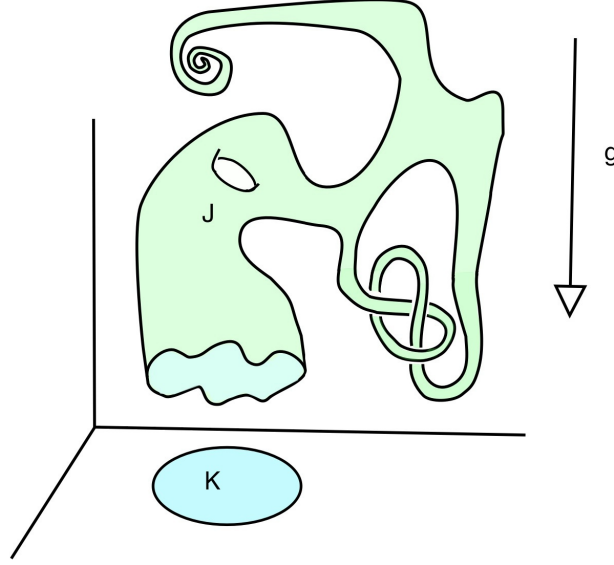


FIGURE 1. General intermediate value theorem

Corollary 4.1.2. *Suppose $\partial J = 0$ and $J \in \hat{\mathcal{B}}_k(U)$ is supported in a contractible open set U . Then $KJ = L$ if and only if $J = \partial L$.*

Proof. Suppose $KJ = L$. Then $\partial KJ = \partial L$. But $J = (K\partial + \partial K)J = \partial KJ = \partial L$. Conversely, suppose $J = \partial L$. Then $(K\partial + \partial K)J = \partial KJ = \partial L$. By Theorem 4.1.1 $KJ = L$. \square

5. DISCRETE POINCARÉ LEMMA

Discrete versions of the Poincaré Lemmas are readily available since Dirac chains are dense in the space of differential chains. If $J \in \hat{\mathcal{B}}_k(U)$, we can approximate J with $A = \sum (p_i; \alpha_i)$ and apply the operator K to each k -element $(p_i; \alpha_i)$. Since K is linear and continuous, KJ is approximated by $\sum K(p_i; \alpha_i)$. The differential complex $\mathcal{A}_k^j(U)$ of Dirac chains of arbitrary order and dimension in U (see [?] §3.4) is the discrete analogue of $\hat{\mathcal{B}}_k(U)$. Since the other operators we use, e.g., boundary, are also linear and continuous, we obtain discrete versions of all of the results of this paper. If we fix a finite set of “base points” $\{p_i\}$, the space of Dirac chains becomes finite dimensional and the operators can be represented as matrices.